# Geometric Reasoning

Measurement and Geometry ➡ **Geometric Reasoning**

★ Formulate proofs involving congruent triangles and angle properties.
  - apply an understanding of relationships to deduce properties of geometric figures (for example the base angles of an isosceles triangle are equal).
★ Apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes.
  - distinguish between a practical demonstration and a proof (for example demonstrating triangles are congruent by placing them on top of each other, as compared to using congruence tests to establish that triangles are congruent).
  - perform a sequence of steps to determine an unknown angle giving a justification in moving from one step to the next.
  - communicate a proof using a sequence of logically connected statements.
★ Prove and apply angle and chord properties of circles.

10A
  - perform a sequence of steps to determine an unknown angle or length in a diagram involving a circle, or circles, giving a justification in moving from one step to the next.
  - communicating a proof using a logical sequence of statements.
  - proving results involving chords of circles.

How do you draw a circle through the three vertices of a triangle?
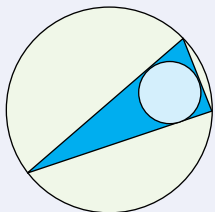
How do you draw a circle to fit inside a triangle?

## A TASK
Find the centre of a triangle:
- Brainstorm different ways of finding the centre of a triangle.
- Use the Internet to research different triangle centres.
- Make large models of various centres of the same triangle.
- Post the models in your classroom.
- Brainstorm practical uses of the various triangle centres.

A LITTLE BIT OF HISTORY
Leonard Euler (1707-1783) made a massive contribution to mathematics.

For example, Euler laid the foundation of analytical mechanics, introduced the notations:
- f(x) for a function,
- e for the base of natural logs,
- i for the square root of -1,
- π for pi,
- $\sum$ for summation.

20   DDR
e-k+f-2
LEONHARD EULER 1707-1783
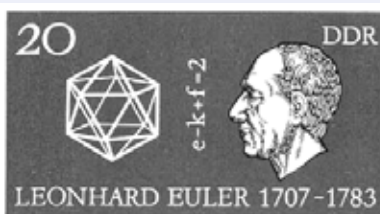
Euler's Formula:
For a simple polyhedron: $F - E + V = 2$
F is the number of faces, E is the number of edges, V is the number of vertices.

Lines
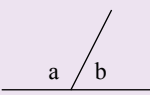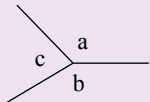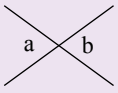
Axioms

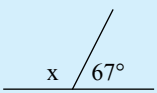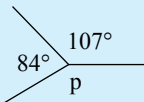An axiom is a statement that is simply accepted as being true.

Deductive Reasoning

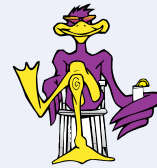Deductive reasoning involves using given true premises to reach a conclusion that is also true.

Figure	Axioms	
	The sum of the angles on a straight line is $180°$	$a + b = 180°$
	The sum of the angles at a point is $360°$	$a + b + c = 360°$
	Vertically opposite angles are equal	$a = b$

Exercise 17.1
Find the value of the unknowns. Show all working:

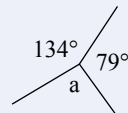> Start with an axiom and use deductive reasoning to reach a conclusion.

$x + 67 = 180$
$x = 180 - 67$
$x = 113°$

$p + 84 + 107 = 360$
$p + 191 \quad = 360$
$p = 360 - 191$
$p = 169°$

1

$x \quad 73°$

2

$68° \quad y$

3

$114°$
a

4

d
$61°$

5

$82° \quad x$
$125°$

6

$134° \quad 79°$
a

7

$y \quad 72°$
$203°$

8

$54°$
$x \quad x$
$118°$

9

$58° \quad x$

10

y
$102°$

11

$137°$
n
m

12

$c \quad d \quad 46°$
$b \quad a \quad 39°$
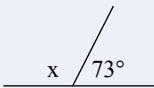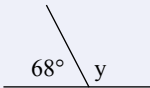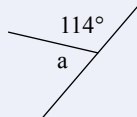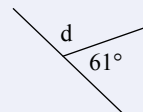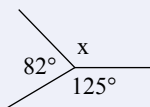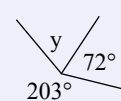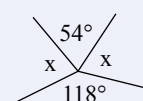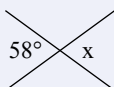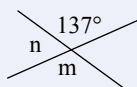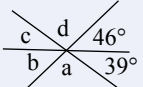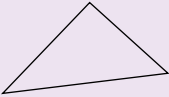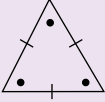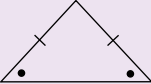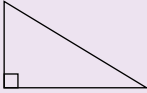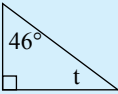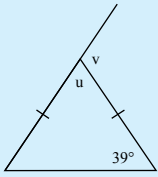
Triangles

Triangle		Axioms
	Scalene triangle	No sides equal Sum angles = 180°
	Equilateral triangle	Each side equal Each angle = 60°
	Isosceles triangle	Two sides equal Two angles equal
	Right-angled triangle	one angle is 90°
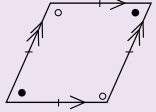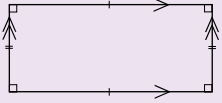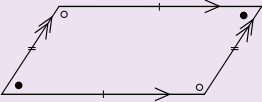
Exercise 17.2

Find the value of the unknowns. Show all working:

$$t + 46 + 90 = 180$$
$$t + 136 = 180$$
$$t = 180 - 136$$
$$t = 44°$$

$$u + 39 + 39 = 180$$
$$u + 78 = 180$$
$$u = 180 - 78$$
$$u = 102°$$

$$v + 102 = 180$$
$$v = 180 - 102$$
$$v = 78°$$

1

43°, x

2
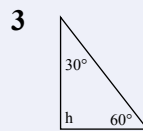
d, 51°

3

30°, h, 60°

4

c

5

x, 24°, 66°

6

b, 37°, 80°

7

35°, 26°, a

8

49°, n

9

88°, x

10

w, 45°

11

b, a, 63°

12

d, 128°, c
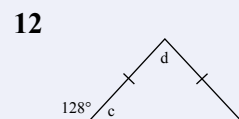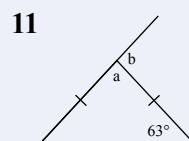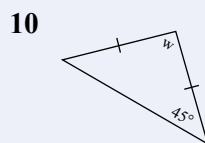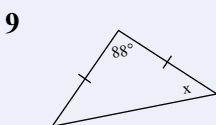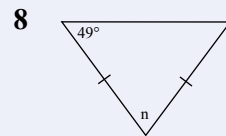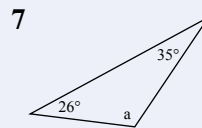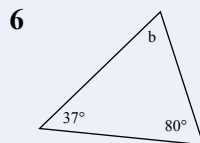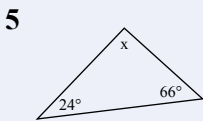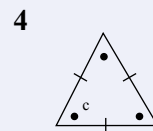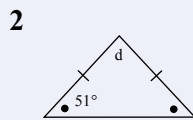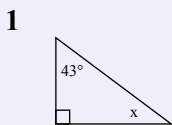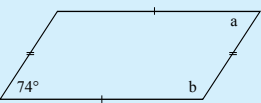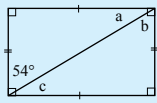
Quadrilaterals

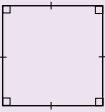| Quadrilateral	Name	Axioms
	Square	Each side equal Each angle = 90°
	Rhombus	Each side equal Opposite angles equal Opposite sides parallel
	Rectangle	Opposite sides equal Each angle = 90° Opposite sides parallel
	Parallelogram	Opposite angles equal Opposite sides equal Opposite sides parallel

Exercise 17.3

Find the value of the unknowns. Show all working:

$a = 74°$ {opposite angles}

$2b + 2 \times 74 = 360$ {360° in quadrilateral }

$2b + 148 = 360$

$2b = 360 - 148$

$2b = 212$

$b = 212 \div 2$

$b = 106°$

Check: $2a + 2b = 360°$

$2 \times 74 + 2 \times 106 = 148 + 212 = 360°$ ✓

$c + 54 = 90$

$c = 90 - 54$

$c = 36°$

$a + 54 + 90 = 180$ {180° in Δ }

$a + 144 = 180$

$a = 180 - 144$

$a = 36°$
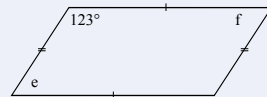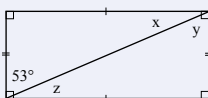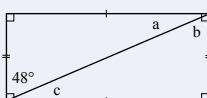
$a + b = 90$

$36 + b = 90$

$b = 90 - 36$

$b = 54°$

1

59°   a   b

2

64°   x   y

3

123°   f   e

4

x   y   53°   z

5

a   b   48°   c

6

y   58°   x   z

Parallel Lines

Angles in parallel lines	Name	Axioms
$a = b$	Alternate angles	Alternate angles are equal
$a = b$	Corresponding angles	Corresponding angles are equal
$a + b = 180°$	Cointerior angles	Cointerior angles sum to 180°

Exercise 17.4
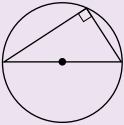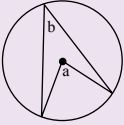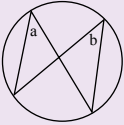
Find the value of the unknowns. Show all working:

$$b = 71° \quad \{Z \text{ ie alternate}\}$$
$$a = b \quad \{F \text{ ie corresponding}\}$$
$$a = 71°$$
$$c + 71 = 180 \quad \{U \text{ ie cointerior}\}$$
$$c = 180 - 71$$
$$c = 109°$$

There is often more than one way to solve these problems eg.,
$$a = 71 \quad \{\text{vertically opposite}\}$$
$$b + c = 180° \quad \{\text{angles straight line}\}$$

1 49° c, a b

2 52° f, d e

3 g 131°, h i

4 j 122°, k l

5 m n o, p 114°

6 q r s, t 129°

7 w v, 123° u

8 67° d, e 73°, f

9 g, h i, 63°, j k

10 56°, x, 76°

11 i, 39° m

12 o p, n 135°, q 128°
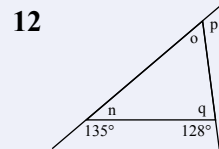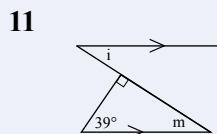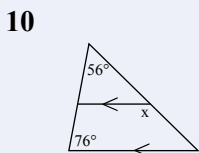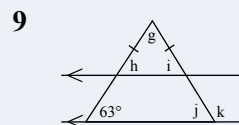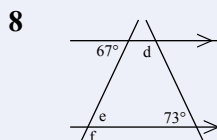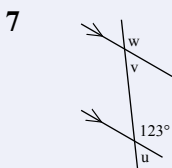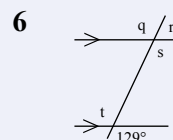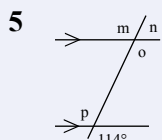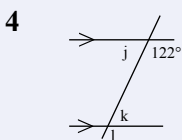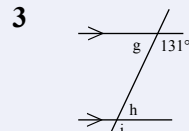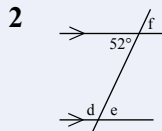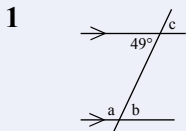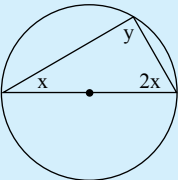
Circles

Angles in a circle	Description	Axioms
	Angles in a semicircle	The angle in a semicircle is 90°
	Central angles and angles on the circle	The angle at the centre is twice the angle on the circle $a = 2b$
	Angles on the same arc	Angles on a circle subtended by the same arc are equal $a = b$

Exercise 17.5 10A

Find the value of the unknowns. Show all working:

$$\underline{y = 90} \qquad \{\text{semicircle angle}\}$$
$$x + 2x + 90 = 180 \qquad \{180° \text{ in a triangle}\}$$
$$3x + 90 = 180$$
$$3x = 180 - 90$$
$$3x = 90$$
$$x = 90 \div 3$$
$$\underline{x = 30°}$$
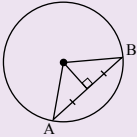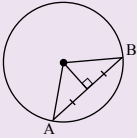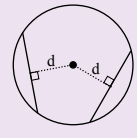
There is an isosceles triangle here. Can you find it?

1

2

3

4

5

6

7

8

9
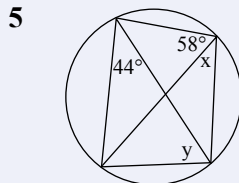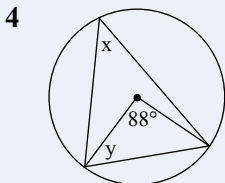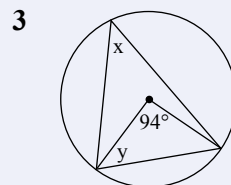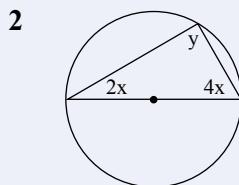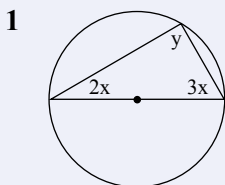
Circles

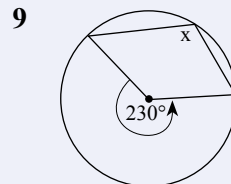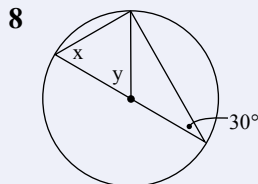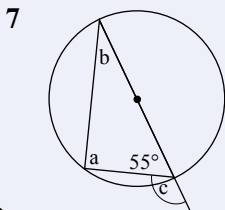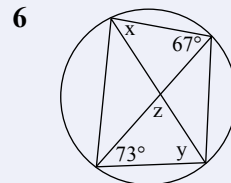| Chords in a circle	Description	Axioms
	Chord bisector (Chord AB)	The perpendicular from the centre to a chord bisects the chord
	Chord perpendicular	The line from the centre to the midpoint of a chord is perpendicular to the chord
	Equal chords	Congruent chords of a circle are the same distance from the centre and subtend equal angles at the centre

Exercise 17.6 10A

Find AB given that CD $\perp$ AB, radius = 35 cm, and CD = 15 cm
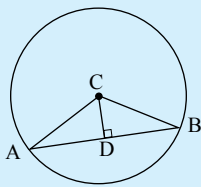
$$CD^2 + DB^2 = CB^2 \quad \text{\{Pythagoras\}}$$
$$15^2 + DB^2 = 35^2$$
$$DB^2 = 35^2 - 15^2$$
$$DB = \sqrt{1000}$$
$$DB = 31.62$$
$$\underline{AB = 63.24 \text{ cm}} \quad \text{\{chord bisector\}}$$

'I never knew what the word round meant until I saw Earth from space' - Aleksei Leonov.

1 Find AD given that CD $\perp$ AB, radius = 59 cm, AB = 45 cm
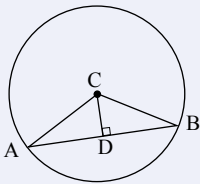
2 Find EG given that CF $\perp$ EG, radius = 11.4 m, FG = 7.3 m
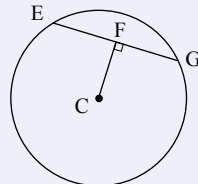

3 Find MN given that CN $\perp$ OM, radius = 30 cm, CN = 20 cm.
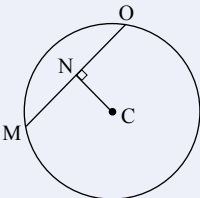
4 Find x and y given that EF = EG,
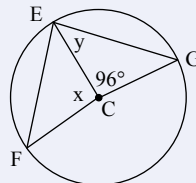
5 Prove that when two circles intersect, that the line line joining their centres bisects the common chord, AB, at right angles.
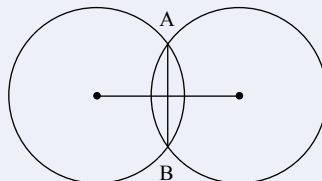
Similarity

Same shape but different size. Sound similar?

Similar triangles have **exactly the same shape** but not necessarily the same size.
- The corresponding **angles are equal**.
- The corresponding sides have the same scale factor.
- The symbol for similarity is ~ .

Two triangles are similar if:

► The three matching angles are equal.   **AAA**
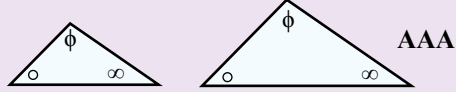
► The three matching sides are in the same ratio   **SSS**
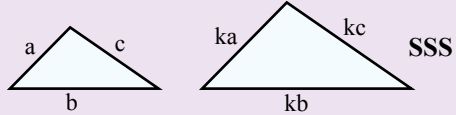
► Two matching sides are in the same ratio and the *included* angles are equal   **SAS**
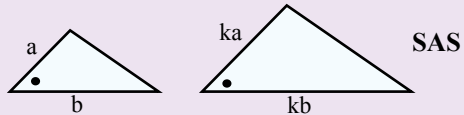
Two right-angled triangles are similar if:

► The hypotenuse and a matching side are in the same ratio   **RHS**
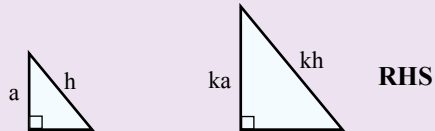
Exercise 17.7

Prove that △ABD ~ △ECD and find BD

EC // AB
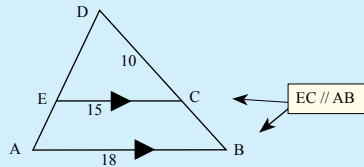
$$\angle D = \angle D \qquad \{common\ angle\}$$
$$\angle DAB = \angle DEC \qquad \{corresponding\ angles\}$$
$$\angle DCE = \angle DBA \qquad \{corresponding\ angles\}$$
$$\therefore \triangle ABD \sim \triangle ECD \qquad \{AAA\}$$

$$\frac{BD}{CD} = \frac{AB}{EC} \quad \{same\ scale\ factor\}$$
$$\frac{BD}{10} = \frac{18}{15}$$
$$BD = \frac{18}{15} \times 10 \qquad \therefore \underline{BD = 12}$$

1 Prove that △ABD ~ △ECD and find DC
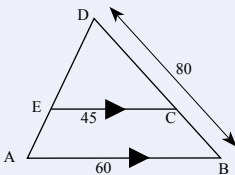
2 Prove that △ACE ~ △BCD and find CD

236
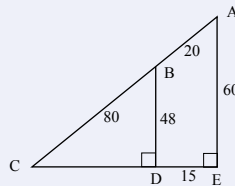
Proofs

A theorem is a statement that can be proved using deductive reasoning.
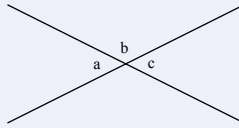
$$a + b = 180° \quad \{180° \text{ on a straight line}\}$$
$$c + b = 180° \quad \{180° \text{ on a straight line}\}$$
$$\therefore a = c$$

Deductive reasoning

Theorem → Vertically opposite angles are equal.

Exercise 17.8

Prove each of the following theorems:

The line joining the midpoints of two sides of a triangle is half the length of the third side.

$$\angle C = \angle C \quad \{\text{common angle}\}$$
$$\frac{CD}{CA} = \frac{1}{2} \quad \{\text{midpoint}\}$$
$$\frac{CE}{CB} = \frac{1}{2} \quad \{\text{midpoint}\}$$
$$\therefore \triangle CDE \sim \triangle CAB \quad \{\textbf{SAS}\}$$

$$\therefore \frac{DE}{AB} = \frac{1}{2} \quad \{\text{same scale factor}\}$$
$$DE = 0.5AB$$

1   A line from the midpoint of a side of a triangle and parallel to another side, bisects the third side.
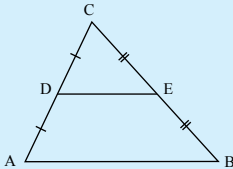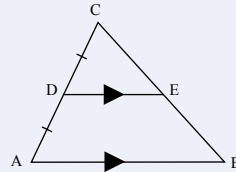

AD = DC
DE // AB
Show that CE = EB
or that $\frac{CE}{CB} = \frac{1}{2}$

2   The line joining the midpoints of two sides of a triangle is parallel to the third side.
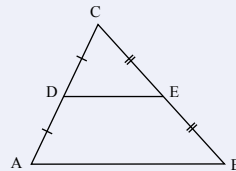
CD = DA
CE = EB
Show that DE // AB
or that $\angle CDE = \angle A$

3   The line from the vertex of an isosceles perpendicular to the base bisects the base.
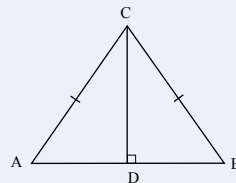
CA = CB
$\angle CDA = \angle CDB$
Show that AD = DB

4   If the line from one angle in a triangle is a perpendicular bisector of the opposite side, then the triangle is an isosceles triangle.
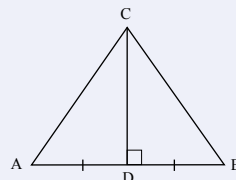
AD = BD
$\angle ADC = \angle BDC$
Show that AC = BC

Mental Computation

Most everyday problems are solved mentally by adults.

Exercise 17.9

1 Spell Equilateral
2 Name two properties of an equilateral triangle.
3 Name two properties of a square
4 If the angle sum of a polygon = $(n−2)×180°$, what is the angle sum of a quadrilateral?
5 What is the gradient of the line $y = 2x + 1$?
6 What is the gradient of the line perpendicular to $y = 2x − 3$?
7 What angle does the line $y = x + 2$ make with the x-axis?
8 Is $x − 1$ a factor of $x^3 − x^2 + x − 1$?
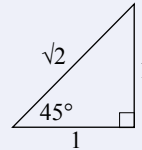9 In the triangle, what is cos 45°?
10 Two sides of a right-angled triangle are 1 and 1, what is the hypotenuse?

$(n−2)×180$
$= (4−2)×180$
$= 360$
Angle sum of a quadrilateral is 360°

$\sqrt{2}$ 1   45°   1

If two lines are perpendicular then the product of their gradients is -1.

$$m_1×m_2 = {}^-1$$

It's zero degrees now and it is predicted to be twice as cold later. How cold will it be?

Exercise 17.10

1 Spell Isosceles
2 Name two properties of an isosceles triangle.
3 Name two properties of a rectangle
4 If the angle sum of a polygon = $(n−2)×180°$, what is the angle sum of a pentagon?
5 What is the gradient of the line $y = 3x + 2$?
6 What is the gradient of the line parallel to $y = x − 1$?
7 What angle does the line $y = {}^-x + 2$ make with the x-axis?
8 Is $x − 2$ a factor of $x^2 − x − 1$?
9 In the triangle, what is sin 60°?
10 Two sides of a right-angled triangle are 1 and 2, what is the hypotenuse?

2   30° $\sqrt{3}$ 60°   1
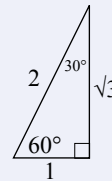
Exercise 17.11

1 Spell Parallelogram
2 Name two properties of a rhombus.
3 Name two properties of a parallelogram
4 If the angle sum of a polygon = $(n−2)×180°$, what is the angle sum of a hexagon?
5 What is the gradient of the line $y = 5x + 3$?
6 What is the gradient of the line perpendicular to $y = 5x − 1$?
7 What angle does the line $y = 5$ make with the x-axis?
8 Is $x − 3$ a factor of $x^2 − 2x − 3$?
9 In the triangle, what is cos 30°?
10 Two sides of a right-angled triangle are 1 and 3, what is the hypotenuse?

What did the little acorn say when it grew up? Geometry.

2   30° $\sqrt{3}$ 60°   1
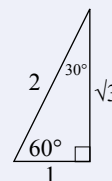
Metallurgists control and develop methods of extracting minerals.
• Relevant school subjects are English, Mathematics, Chemistry, Physics.
• Courses usually involve an engineering degree.

Competition Questions

Build maths muscle and prepare for mathematics competitions at the same time.

Exercise 17.12

Find the value of x in each of the following:

x+30°	Angle sum $= 360°$
x	$x + 30 + x + x - 5 + x + 35 = 360$
	$4x + 60 = 360$
	$4x = 360 - 60$
	$4x = 300$
x−5° x+35°	$x = 300 \div 4$
	$\underline{x = 75°}$

1

2 43° x 34°

3

4 130° x

5 2x 2x x

6 59° 44° x 37°

7 x 133° 107°

8 52° x

9 29° x 22°

10 x 134°

11 x 81° 52°

12 x+40° x x−8° x+44°

13 24° x

14 x

15 64° 83° x
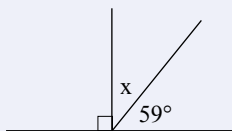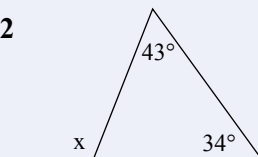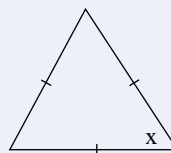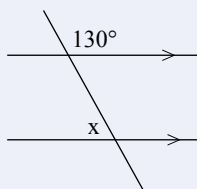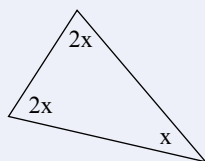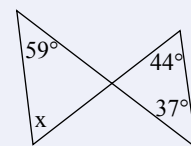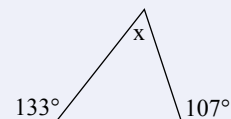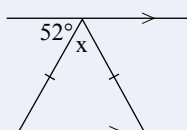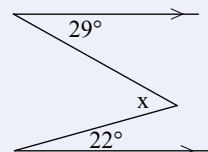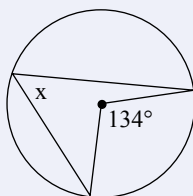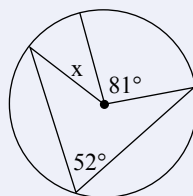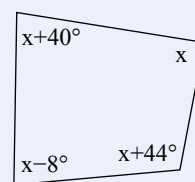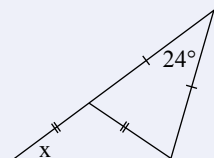
A Couple of Puzzles

Exercise 17.13

1 Fran earns $650 per week and Fred earns $x less per week. Together they earn $1250. Find x.

2 An isosceles triangle has angles of a°, a°, and 30°. Find a.

3 A regular pentagon can be constructed with five isosceles triangles. What is the size of each angle of the isosceles triangle?
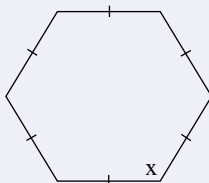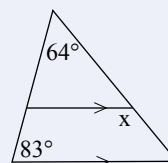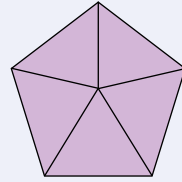
4 Find the numbers that correspond to each letter in the Alphanumerics:

a)		**b)**		**c)**		**d)**	
	WE				LIES		HOME
	ATE		HOCUS		LIES		HOME
	+ WE		+ POCUS		+ ARE		+ TO
	GREW		PRESTO		SILLY		MOMMA

A Game

Fours is a calculator game in which the first person to have a calculator display a selected number is the winner.

1 Randomly select a number from 1 to 100.

> Use the random button on your calculator?
> (100)(×)(Rand)(=)

2 Use only the 4, +, −, x, = keys and the y^x or ^ key to produce the selected number on the display of your calculator.

A Sweet Trick

1	Think of any number from 1 to 100.	92
2	Write down the name of the number.	Ninety-two
3	Count the number of letters in the number.	9
4	Write the name of the number.	nine
5	Count the number of letters in the number.	4
6	Write the name of the number.	four
7	Count the number of letters in the number.	4
8	Continue until a number repeats.	four, 4.

> What is the number?

> The number is always 4.

Investigations

Investigation 17.1   Isosceles Triangles

1   Draw the following triangles:
   a)   AB=10 cm, AC=10 cm, BC=5 cm
   b)   AB=10 cm, AC=10 cm, BC=10 cm
   c)   AB=10 cm, AC=10 cm, BC=15 cm

2   Measure the angles in each triangle and complete a table similar to the following:
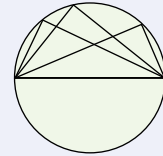
3   What relationships do you notice?

	$\angle A$	$\angle B$	$\angle C$
a)			
b)			
c)			

Investigation 17.2   Angles in a semicircle

1   Draw a circle of radius 10 cm.

2   Draw a diameter.

3   Draw a number of angles in the semicircle. similar to the example shown.

4   Measure the angles on the semicircle. What do you notice?

**5**   Use scissors to cut out an interesting arrangement and post in your classroom.

Investigation 17.3   Central angles and angles on the circle

1   Draw a circle of radius 10 cm.

2   From an arc draw the central angle (to the centre of the circle).

3   From the same arc draw a number of angles on the circle similar to the example shown.

4   Measure the central angle and the angles on the circle. What do you notice?

5   Use scissors to cut out an interesting arrangement and post in your classroom.
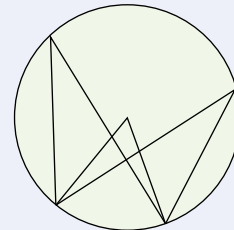
Investigation 17.4

1   Construct a triangle ABC in which AB is 10 cm long.

2   Mark a point D on AB such that AD = 5 cm.

3   Draw DE parallel to BC, E being a point on AC.

4   Measure the lengths of AE and EC.
   a)   What is the ratio of AD to DB?
   b)   What is the ratio of AE to EC?

5   Repeat with the following:

AB	12	12	12	16
AD	6	4	9	12

Technology

Technology 17.1

Start computer programming. The LOGO computer language is a great starting point.

1. Find a LOGO site on the Internet (Microworlds, LOGO, turtle).
2. Program the turtle to draw a square
3. Program the turtle to draw a regular hexagon (eg. Forward 10, Right 120, etc).
4. Try to produce some of the shapes described earlier in this chapter.

```
Forward 10
Right 90
Forward 10
Right 90
Forward 10
Right 90
Forward 10
```

Technology 17.2

Produce a Powerpoint slide show of your LOGO programming

1. Use LOGO to produce a shape.
2. Press the Print Screen key on your keyboard.
3. Open Powerpoint, and paste the shape on a slide.

> Fractals are found in nature. They have applications in soil mechanics, seismology, medicine and artwork.

Technology 17.3   Fractals

A fractal is a geometric shape that can be split into parts.
Each part being similar to the original shape.

> Start with an equilateral triangle
> Repeat three times:
>   Add triangles a third
>   the size to each side.

a) Draw the first four iterations of the Koch snowflake'

> Each iteration produces smaller similar shapes.

Use Internet software to draw iterations of the Koch Snowflake.
Use search phrases such as 'Koch Snowflake' with 'applet', 'interactive' etc.

Technology 17.4   Fabulous Fern Fractals

> **Fern Fractals**
>
> Watch a Barnsley Fern fractal video.

> **Fern Fractals**
>
> Form fabulous fern fractals.

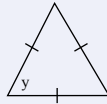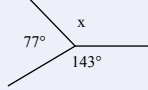## Exercise 17.14

1 Find the value of the unknowns. Show all working:

a)

59° x

b)

156° 109°
y

c)

72°
z 63°
133°

d)

c 91° 45°
b a 37°

e)

43°
x

f)

y

g)

93°
x

h)

y
x
46°

i)

a
57° b

j)

x y
68°
z

k)

y 69°
x
z

l)

x
63°
z y

m)

a
b c
55° d e

n)

54°
x
81°

o) 10A

y
2x 3x

p) 10A

x
96°
y

q) 10A

b
a 58°
c
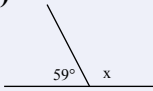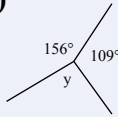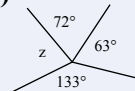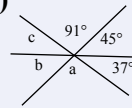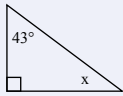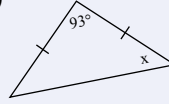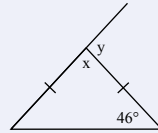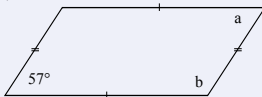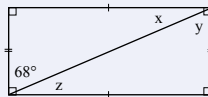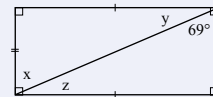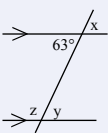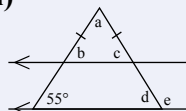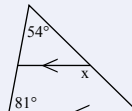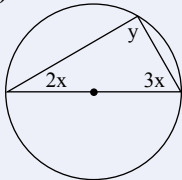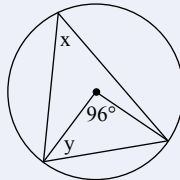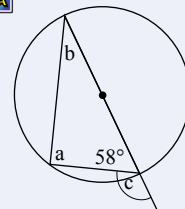
2 Prove that △ABD ~ △ECD and find DC

D
80
E 45 C
A 60 B
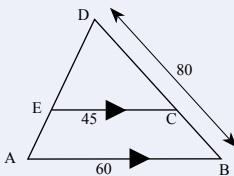
3 Prove that the line from the midpoint of a side of a triangle and parallel to another side, bisects the third side.

C
D E
A B
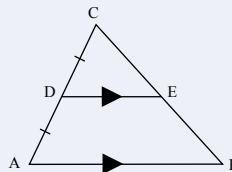
4 10A Find x and y given that EF = EG,
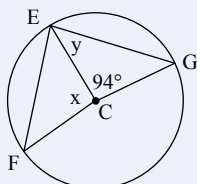
E
y G
94°
x C
F

Deductive Reasoning

Deductive reasoning involves using given true premises to reach a conclusion that is also true.

Exercise 17.15

1   Find the value of the unknowns. Show all working:

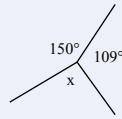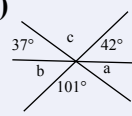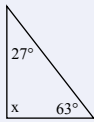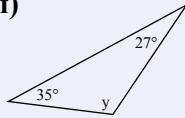**a)**

x $65°$

b)

x
$77°$ $143°$

c)

$150°$ $109°$
x

d)

$37°$ c $42°$
b a
$101°$

e)

$27°$
x $63°$

f)

$27°$
$35°$ y

g)

$91°$
z

h)

b
$134°$ a c

i)

$120°$ y
x

j)

x y
$67°$ z

k)

y $65°$
x z

l)

x
$61°$
z y

m)

a
b c
$48°$ d e

n)

$47°$
x
$75°$

o) 10A

y
$2x$ $4x$

p) 10A

x
$93°$
y

q) 10A

x
y
$28°$
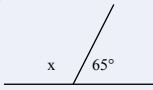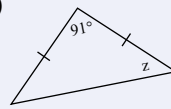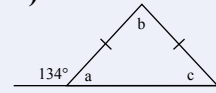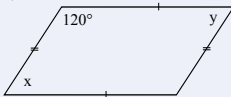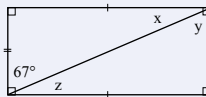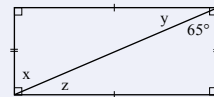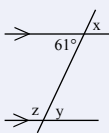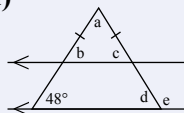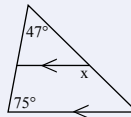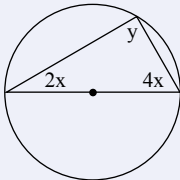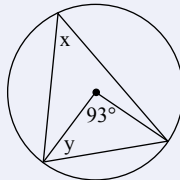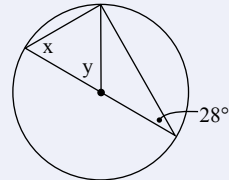
2   Prove that △ACE ~ △BCD and find DE

A
25
B
60
50   40
C
30   D   E
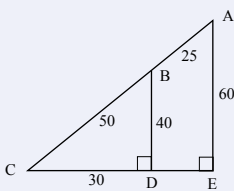
3   Prove that the line joining the midpoints of two sides of a triangle is parallel to the third side.

C
D   E
A   B
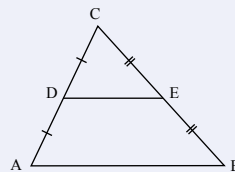
4   10A Find MO given that CN ⊥ OM, radius = 33 cm, CN = 21 cm.

O
N
C
M

A theorem is a statement that can be proved using deductive reasoning.